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THE ANISOTROPIC THERMOMECHANICAL CONSTITUTIVE THEORY FOR A FLUID-FILLED POROUS MATERIAL WITH SOLID/FLUID OUTER BOUNDARY

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Abstract—The anisotropic thermomechanical constitutive theory for a fluid-filled porous material with a solid/fluid outer boundary is constructed along the same lines as in the author's previous work. The notion that a representative volume element may be considered as a superposition of the porous solid material and the porous fluid material is employed. The constitutive theory with a uniform temperature increase is constructed by use of superposition. The thermal effective stress laws for various kinematical quantities are obtained.

1. INTRODUCTION

References [1–3] constructed the constitutive theory of a fluid-filled porous material from a slightly different point of view from that of Biot[4–6] and gave a further insight into the well-known theory. Most recently, Ref. [7] has pointed out that the outer boundary of the representative volume element consists of a solid outer boundary and a fluid outer boundary, and therefore, careful consideration must be made in defining the kinematical and kinetical quantities. In this work, the anisotropic thermomechanical constitutive theory for a fluid-filled porous material with a solid/fluid outer boundary is constructed along the same lines as in the author's previous work.

The following relations were shown to hold among the volume average strains defined for a sample with a broken outer boundary in Ref. [7]

$$e_{ij}^{(1)} = (1-\phi)e_{ij}^{s}+\phi e_{ij}^{P(1)} \tag{1}$$

$$e_{ij}^{(1)} = e_{ij}^{s}+e_{ij}^{*(1)}, \quad e_{ij}^{*(1)} = \phi(e_{ij}^{P(1)}-e_{ij}^{s}) \tag{2}$$

$$e_{ij}^{(2)} = \phi e_{ij}^{f}+(1-\phi)e_{ij}^{P(2)} \tag{3}$$

$$e_{ij}^{(2)} = e_{ij}^{f}+e_{ij}^{*(2)}, \quad e_{ij}^{*(2)} = (1-\phi)(e_{ij}^{P(2)}-e_{ij}^{f}) \tag{4}$$

$$(1-\phi)e_{ij}^{*(1)}+\phi e_{ij}^{*(2)} = 0. \tag{5}$$

The superscripts (1), (2), s, f, P(1) and P(2), respectively, indicate the porous solid material, the porous fluid material, the solid matrix, the fluid without pores, the pore of the porous solid material and the pore of the porous fluid material, and ϕ is the volumetric porosity of the porous solid material. The total strain components $e_{ij}^{*(1)}$ in eqns (2) and $e_{ij}^{*(2)}$ in eqns (4), respectively, measure the differential straining of the pore space of the porous solid material and the solid matrix, and that of the porous fluid material and the fluid, respectively; and they are related by eqn (5).

Similarly, under the assumption of a stationary solid, non-zero components of the volume average strain rates were obtained as follows:

$$d_{ij}^{(2)} = d_{ij}^{f} = d_{ij}^{P(2)}. \tag{6}$$

The volume average stresses for a sample with solid/fluid outer boundary in Ref. [7]

are summarized in eqns (7)–(13) where the superscripts have the same meanings as those in the kinematical quantities and t_{ij} are the components of the volume average stress of this representative volume element

$$t^{(1)}_{(ij)} = (1-\phi)t^{s}_{ij}+\phi t^{P(1)}_{(ij)} \tag{7}$$

$$t^{(1)}_{[ij]} = \phi t^{P(1)}_{[ij]} \tag{8}$$

$$t^{(2)}_{(ij)} = \phi t^{f}_{ij}+(1-\phi)t^{P(2)}_{(ij)} \tag{9}$$

$$t^{(2)}_{[ij]} = (1-\phi)t^{P(2)}_{[ij]} \tag{10}$$

$$t_{ij} = t^{(1)}_{ij}+t^{(2)}_{ij} \tag{11}$$

$$t_{ij} = (1-\phi)t^{s}_{ij}+\phi t^{f}_{ij} \tag{12}$$

$$\phi t^{P(1)}_{ij}+(1-\phi)t^{P(2)}_{ij} = 0. \tag{13}$$

The stress tensor of the porous solid material and that of the porous fluid material are shown to have a symmetric and a skew-symmetric part, represented by ( ) and [ ], respectively, in eqns (7)–(10). The skew-symmetric stress tensor of the porous solid material is equal in magnitude and opposite in sign to that of the porous fluid material, as in eqns (8), (10) and (13). The skew-symmetric part of the stress tensor of the porous solid material was shown to be related to the interaction moment per unit representative volume element in Ref. [7].

2. CONSTITUTIVE THEORY

The representative volume element is subject to a uniform temperature increase θ, a pointwise stress distribution $t_i(\mathrm{x})$ on the solid outer boundary B and the fluid outer boundary B_{fo}. Within the linear range, the volume average kinetical and kinematical quantities may be related by their local response. Thus, we may write down the constitutive equations for t^{s}_{ij} and t^{f}_{ij} as follows:

$$t^{s}_{ij} = M^{s}_{ijkl}e^{s}_{kl}-M^{s}_{ijkl}B^{s}_{kl}\theta \tag{14}$$

$$P_f = -K^f e^f_{mm}+3\alpha^f K^f\theta \tag{15}$$

$$t^f_{ij}+P_f\delta_{ij} = \lambda^f d^f_{mm}\delta_{ij}+2\mu^f d^f_{ij} \tag{16}$$

where M^{s}_{ijkl} and B^{s}_{ij}, and K^f and α^f, are the elastic moduli and the thermal expansion coefficient of the solid, and the bulk modulus and the thermal expansion of the fluid, respectively.

In an attempt to relate other kinetical and kinematical quantities, we separate the total loading on the solid matrix into loading conditions (17) and (18), and that on the fluid into loading conditions (19) and (20), respectively

$$\begin{aligned} t_i &= t_i(\mathrm{x}) \quad \text{on } B_{so} \\ t_i &= -P_f n_i \quad \text{on } B_i \end{aligned} \tag{17}$$

a constant temperature increase θ

$$t_i = 0 \quad \text{on } B_{so}$$
$$t_i = t_i(\mathrm{x}) + P_f n_i \quad \text{on } B_i \tag{18}$$

$$t_i = -P_f n_i \quad \text{on } B_{fo}$$
$$t_i = -P_f(-n_i) \quad \text{on } B_i \tag{19}$$

a constant temperature increase θ

$$t_i = t_i(\mathrm{x}) + P_f n_i \quad \text{on } B_{fo}$$
$$t_i = t_i(\mathrm{x}) + P_f(-n_i) \quad \text{on } B_i \tag{20}$$

where P_f is the volume average hydrostatic fluid pressure and a constant. The unit normal vector components on the inner boundary B_i are positive in the outward direction on the solid inner boundary.

The purely mechanical constitutive theory for a fluid-filled porous material such as that of Biot[4–6] and that given in Refs [1–3, 7] were constructed under the assumption that the pointwise stress distribution on the pore boundary of the porous solid material may be approximated by the volume average hydrostatic fluid pressure P_f when the deformations of the porous solid material were considered. Employing the same assumptions, we assume that loading conditions (18) and (20) cause zero volume average strains. Then eqns (7)–(13) lead to eqns (21)–(25) in addition to eqns (7), (8) and (13). In eqn (21), the total volume average stress is decomposed into the part related to the deformation t'_{ij} and that related to the fluid flow t''_{ij}

$$t_{ij} = t'_{ij} + t''_{ij} = (1-\phi)t^s_{ij} + \phi t^f_{ij} \tag{21}$$

$$t'_{ij} = (1-\phi)t^s_{ij} - \phi P_f \delta_{ij} \tag{22}$$

$$t'_{ij} = t^{(1)}_{ij} + t^{(2)\prime}_{ij} \tag{23}$$

$$t^{(2)\prime}_{ij} = -\phi P_f \delta_{ij} + (1-\phi)t^{P(2)}_{ij} \tag{24}$$

$$t''_{ij} = t^{(2)\prime\prime}_{ij} = \phi(t^f_{ij} + P_f \delta_{ij}). \tag{25}$$

The loading, eqns (17), may be decomposed into loadings (26) and (27)

$$t_i = t^s_{ij} n_j \quad \text{on } B_{so}$$
$$t_i = t^s_{ij} n_j \quad \text{on } B_i \tag{26}$$

a constant temperature increase θ

$$t_i = t_i(\mathrm{x}) - t^s_{ij} n_j \quad \text{on } B_{so}$$
$$t_i = -t^s_{ij} n_j - P_f n_i \quad \text{on } B_i. \tag{27}$$

Under loading condition (26), the stress and strain states in the solid matrix of the porous solid material are uniform and the total strain of a porous solid material is e^s_{ij}. From eqns (2), loading condition (27) will cause a total strain of a porous solid material $e^{*(1)}_{ij}$. In the purely mechanical case[7], loading condition (27) is shown to lead to the constitutive equation, eqn (28), by the methods of superposition of loadings, where the components of the elastic compliance of the porous solid material are denoted by C_{ijkl}

$$e^{*(1)}_{ij} = C^{*(1)}_{ijkl}(t'_{kl} + P_f \delta_{kl}) \tag{28}$$

$$C^{*(1)}_{ijkl} = C_{ijkl} - \frac{1}{1-\phi} C^{s}_{ijkl}. \tag{29}$$

We obtain the alternative constitutive equations, eqns (30) and (31), by rewriting eqns (14), (15) and (28) with the use of eqns (22), (2), (4) and (5)

$$\begin{aligned} t^{s}_{ij} = {} & M^{s}_{ijkl}[\tfrac{1}{2}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) - (1-\phi)C^{a}_{klrs}M^{s}_{rsmn}]e^{(1)}_{mn} \\ & + (1-\phi)K^{f}M^{s}_{ijkl}C^{a}_{klrr}e^{(2)}_{ss} - (1-\phi)(3\alpha^{f}K^{f}M^{s}_{ijkl}C^{a}_{klmn} - M^{s}_{ijkl}C^{a}_{klmn}M^{s}_{mnrs}B^{s}_{rs})\theta \end{aligned} \tag{30}$$

$$\begin{aligned} P_f = {} & -\frac{(1-\phi)^2}{\phi}K^{f}C^{a}_{mmij}M^{s}_{ijkl}e^{(1)}_{kl} - K^{f}\left[1 - \frac{(1-\phi)^2}{\phi}K^{f}C^{a}_{mmii}\right]e^{(2)}_{rr} \\ & + K^{f}\left[\frac{3(1-\phi)^2}{\phi}\alpha^{f}K^{f}C^{a}_{mmii} + 3\alpha^{f} - \frac{(1-\phi)^2}{\phi}C^{a}_{mmij}M^{s}_{ijkl}B^{s}_{kl}\right]\theta \end{aligned} \tag{31}$$

where

$$\begin{aligned} M^{a}_{ijkl} &= M^{*(1)}_{ijkl} + (1-\phi)M^{s}_{ijkl} + \frac{(1-\phi)^2}{\phi}K^{f}\delta_{ij}\delta_{kl} \\ C^{a}_{ijkl}M^{a}_{klrs} &= \tfrac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}). \end{aligned} \tag{32}$$

When the fluid is incompressible, we have eqn (33) instead of eqn (15), and obtain the corresponding alternative constitutive equations, eqns (34) and (35)

$$e^{f}_{mm} = 3\alpha^{f}\theta \tag{33}$$

$$\begin{aligned} t^{s}_{ij} = {} & M^{s}_{ijkl}C^{b}_{klrs}\left[M^{*(1)}_{rsmn} + \frac{(1-\phi)\delta_{rs}C^{b}_{qqot}M^{s}_{otmn}}{C^{b}_{uuvv}}\right]e^{(1)}_{mn} \\ & + \frac{\phi}{(1-\phi)}\frac{M^{s}_{ijkl}C^{b}_{klrr}}{C^{b}_{uuvv}}e^{(2)}_{ss} + (1-\phi)M^{s}_{ijkl}C^{b}_{klrs} \\ & \times\left[M^{s}_{rsmn}B^{s}_{mn} - \frac{\delta_{rs}}{C^{b}_{uuvv}}\left(C^{b}_{qqot}M^{s}_{otmn}B^{s}_{mn} + \frac{3\alpha^{f}\phi}{(1-\phi)^2}\right)\theta\right] \end{aligned} \tag{34}$$

$$P_f = -\frac{C^{b}_{mmij}M^{s}_{ijkl}}{C^{b}_{nnpp}}e^{(1)}_{kl} - \frac{\phi}{(1-\phi)^2 C^{b}_{nnpp}}e^{(2)}_{mm} + \left(\frac{C^{b}_{mmij}M^{s}_{ijkl}B^{s}_{kl}}{C^{b}_{nnpp}} + \frac{3\alpha^{f}\phi}{(1-\phi)^2 C^{b}_{nnpp}}\right)\theta \tag{35}$$

where

$$\begin{aligned} M^{b}_{ijkl} &= M^{*(1)}_{ijkl} + (1-\phi)M^{s}_{ijkl} \\ C^{b}_{ijkl}M^{b}_{klrs} &= \tfrac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}). \end{aligned} \tag{36}$$

In the purely mechanical case[7], the constitutive equations for a sample with solid/fluid outer boundary are shown to take the same form as those for a sample with closed boundary, while the definitions of volume average kinetical and kinematical quantities are different. This is true when we include the temperature effect in the formulations. Therefore, when both the solid material and the pore structure are isotropic, the constitutive equations, eqns (30) and (31) and (34) and (35) reduce to those in Ref. [3], where the isotropic thermomechanical constitutive equations for a sample with a closed outer boundary are recorded.

As in the purely mechanical case[7], the constitutive equation for the flow of the porous fluid material is given by

$$t^{f}_{ij} + P_f\delta_{ij} = \lambda^f d^{(2)}_{mm}\delta_{ij} + 2\mu^f d^{(2)}_{ij}. \tag{37}$$

The constitutive equations for the interaction related kinetical quantities such as $t^{(1)}_{[ij]}$ may not be obtained by the volume averaging methods.

3. THERMAL EFFECTIVE STRESS LAWS

In eqn (28), it is shown that the strain components of a porous solid material due to differential straining of the pore space and the solid matrix is proportional to Terzaghi's effective stress. Equations (1)–(5), (14), (15), (22), (28) and (29) lead to the following thermal effective stress laws:

$$e^{(1)}_{ij} = C_{ijkl}T'_{kl} \tag{38}$$

$$e^{P(1)}_{ij} = C^{P}_{ijkl}\{t'_{kl}\} \tag{39}$$

$$e^{(2)}_{mm} = -\frac{1-\phi}{\phi}C^{*(1)}_{mmkl}\{\{t'_{kl}\}\} \tag{40}$$

$$\Delta m = \rho_f\phi C'^{P}_{iikl}[[t'_{kl}]] \tag{41}$$

where

$$T'_{kl} = t'_{kl} + (\delta_{kl} - M_{klrs}C^{s}_{rsmm})P_f + M_{klrs}B^{s}_{rs}\theta \tag{42}$$

$$\{t'_{kl}\} = t'_{kl} + (\delta_{kl} - M^{P}_{klrs}C^{s}_{rsmm})P_f + M^{P}_{klrs}B^{s}_{rs}\theta \tag{43}$$

$$\{\{t'_{kl}\}\} = t'_{kl} + \left(\delta_{kl} + M^{*(1)}_{klrr}\frac{\phi}{3K^f(1-\phi)}\right)P_f - \frac{\phi}{1-\phi}M^{*(1)}_{klrr}\alpha^f\theta \tag{44}$$

$$[[t'_{kl}]] = t'_{kl} + \left\{\delta_{kl} + \tfrac{1}{3}M'^{P}_{klmm}\left(\frac{1}{K^f} - C^{s}_{jjnn}\right)\right\}P_f + \tfrac{1}{3}M'^{P}_{klmm}(B^{s}_{rr} - 3\alpha^f)\theta \tag{45}$$

$$C'^{P}_{ijkl} = \frac{1}{\phi}C^{*(1)}_{ijkl} + \frac{1}{1-\phi}C^{s}_{ijkl} \tag{46}$$

Δm in eqn (41) indicates the fluid mass increase per unit volume of the porous solid material and may be expressed as

$$\Delta m = \rho_f\phi(e^{(1)}_{mm} - e^{(2)}_{mm}) \tag{47}$$

where ρ_f is the fluid mass density.

Also by setting $\Delta m = 0$ in eqn (41), we may obtain the undrained response of the total volumetric strain for a hydrostatic confining stress state $t'_{ij} = -P_f\delta_{ij}$ as follows:

$$P - A\theta = -K_u e^{(1)}_{ii} \tag{48}$$

where

$$A = \left\{\tfrac{1}{3}C_{jjkk}M^{\mathrm{p}}_{llmm}\left(\frac{1}{K^{\mathrm{f}}} - C^{\mathrm{s}}_{nnpp}\right) - 3C^{\mathrm{s}}_{jjkk}\right\}$$
$$\times\left[\left\{3+\tfrac{1}{3}M^{\mathrm{p}}_{rrss}\left(\frac{1}{K^{\mathrm{f}}} - C_{ttuu}\right)\right\}B^{\mathrm{s}}_{vv} + M^{\mathrm{p}}_{rrss}\alpha^{\mathrm{f}}(C_{ttuu} - C^{\mathrm{s}}_{ttuu})\right]. \quad (49)$$

In the absence of a temperature increase, the effective stress laws for the purely mechanical case are recovered[1].

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